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MASS OR HEAT TRANSFER WITH A CHANGE IN INTERFACIAL AREA—I

MASS TRANSFER IN THE CONTINUOUS PHASE TO A GROWING DROP

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Abstract—A similarity transformation is used for solving the problem of mass transfer to a growing drop or bubble when not only the radial velocity component is taken into account (as in previous papers) but also the tangential velocity component. The method may be used either when the growth is due to the heat transfer itself or to the flow-rate of a liquid fed through a capillary.

NOMENCLATURE

a, radius of the drop;
A, constant defined by equation (13);
B, constant defined by equation (13);
C, concentration;
D, diffusion coefficient;
E, integration constant;
f, quantity defined by equation (6);
m, liquid flow-rate through the capillary;
N_θ, mass flux;
N, average mass flux with respect to θ;
p, integration variable;
r, radial variable (spherical coordinate system);
Re, Reynolds number, $= apU/\mu$;
s, integration variable;
t, time;
U, $\mathrm{d}a/\mathrm{d}t$;
v_r, radial component of velocity;
v_θ, tangential component of velocity;
y, $r - a$;
α, integration constant;
β, μ_1/μ;
δ, thickness of the diffusion boundary layer;
ε, δ^2;
η, $y/\delta(\theta, t)$;
θ, polar angle (spherical coordinate system);

λ, quantity defined by equation (2);
μ, dynamic viscosity of the continuous phase;
μ_1, dynamic viscosity of the discontinuous phase;
ν, kinematic viscosity of the continuous phase;
ρ, the density of the continuous phase;
τ, time;
φ, quantity defined by equation (7);
χ, quantity defined by equation (8);
ω, constant selected equal to 2.

THERE exist numerous cases in which the interfacial area increases or decreases either as a consequence of the heat and/or the mass transfer or during the transfer process. As examples, one may mention the heat transfer to a growing bubble in boiling heat transfer and the mass transfer to a drop in its forming period. In the first case the growth of the bubble is determined by the heat transfer itself, while in the second by the flow-rate of the liquid fed through the capillary at the tip of which the drop is formed. The first problem was treated by Forster and Zuber [1], by Plesset and Zwick [2] and by Scriven [3]. The second by Ilkovic [4] and by Beek and Kramers [5]. The treatments mentioned have taken into account only the radial velocity component. Recently Golub and Krîlov [6] have treated the problem of mass transfer in the continuous phase to a growing forming drop by taking into account the tangential velocity component too. They have solved the hydrodynamic problem for small Reynolds numbers and have used a perturbation technique for solving the mass-transfer problem. The aim of the present paper is to show that the last problem and other similar problems may be solved by means of a method, developed in [7], based on a similarity transformation. Besides its simplicity, the method does not require that the tangential velocity component should be a small quantity as is required by the perturbation method.

BASIC EQUATIONS

Let us consider a growing drop which is formed at the tip of a capillary owing to the flow-rate m of the liquid. The time dependence of the drop radius results from the equation

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{4}{3}\pi a^3) = m.$$

One obtains that

$$a(t) = \lambda t^{\frac{1}{3}}, \tag{1}$$

where

$$\lambda = \left(\frac{3m}{4\pi\rho}\right)^{\frac{1}{3}}. \tag{2}$$

For the velocity components in the continuous phase the following expressions were obtained by Golub and Krîlov for small Reynolds numbers by means of a perturbation technique [6]

$$v_r = U\left\{f(r, t)\cos\theta + \frac{a^2}{r^2}\right\} \tag{3}$$

$$v_\theta = -\,U\,\varphi(r,t)\sin\theta \tag{4}$$

where

$$U = \frac{\mathrm{d}a}{\mathrm{d}t} \tag{5}$$

$$f = \frac{2y}{3a(2+5\beta)\chi^2}\left[1 - \frac{y}{a}(1 - \tfrac{9}{2}\beta) + \dots\right] - \frac{28y}{15a(2+3\beta)\chi^3}\left[1 - \frac{y}{a}(1 - \tfrac{3}{2}\beta) + \dots\right] + \dots. \tag{6}$$

$$\begin{aligned}\varphi = \frac{1}{3(2+5\beta)\chi^2}&\left[1 + \frac{y}{a}(1+9\beta) - \frac{4y^2}{a^2}(1 - \tfrac{9}{2}\beta) + \dots\right] \\ &+ \frac{14}{15(2+3\beta)\chi^3}\left[1 + \frac{y}{a}(1+3\beta) - \frac{4y^2}{a^2}(1 - \tfrac{3}{2}\beta) + \dots\right] +\end{aligned} \tag{7}$$

$$\chi = \lambda^{-1}(\nu^3 t)^{\frac{1}{6}} \equiv (3Re)^{-\frac{1}{2}}, \qquad \beta = \frac{\mu_1}{\mu}. \tag{8}$$

Neglecting as usual a number of terms in the convective diffusion equation, one may write

$$\frac{\partial c}{\partial t} + v_r\frac{\partial c}{\partial r} + \frac{v_\theta}{r}\frac{\partial c}{\partial \theta} = D\frac{\partial^2 c}{\partial r^2}. \tag{9}$$

The change of variables

$$\begin{aligned} y &= r - a \\ \tau &= t \end{aligned} \tag{10}$$

allows to write equation (9) under the form

$$\frac{\partial c}{\partial \tau} + \left(v_r - \frac{\mathrm{d}a}{\mathrm{d}t}\right)\frac{\partial c}{\partial y} + \frac{v_\theta}{a}\frac{\partial c}{\partial \theta} = D\frac{\partial^2 c}{\partial y^2}. \tag{11}$$

The diffusion coefficient being small, the depth of penetration by diffusion is small too. The region of interest for the concentration being near the interface where $y \ll a$ one may approximate the velocity distribution by means of the expressions containing only the first term of the series expansion with respect to y/a.

In this manner one obtains

$$\frac{\partial c}{\partial t} + \frac{2y}{3}\left[(At^{-\frac{4}{3}} + Bt^{-\frac{3}{2}})\cos\theta - t^{-1}\right]\frac{\partial c}{\partial y} - \tfrac{1}{3}(At^{-\frac{4}{3}} + Bt^{-\frac{3}{2}})\sin\theta\frac{\partial c}{\partial \theta} = D\frac{\partial^2 c}{\partial y^2} \tag{12}$$

where

$$A = \frac{1}{3(2+5\beta)\lambda^{-2}\nu}; \qquad B = \frac{14}{15(2+3\beta)\lambda^{-3}\nu^{\frac{3}{2}}}. \tag{13}$$

Equation (12) must be solved for the boundary conditions:

$$\begin{aligned} c &= c_0 \quad \text{for} \quad t = 0 \\ c &= c_i \quad \text{for} \quad y = 0 \\ c &= c_0 \quad \text{for} \quad y \to \infty. \end{aligned} \tag{14}$$

THE SOLUTION OF EQUATION (12)

Since the coefficient multipying $\partial c/\partial y$ is proportional to y, while that multiplying $\partial c/\partial \theta$ is independent on y it is possible to use the similarity variable η [7]

$$\eta = \frac{y}{\delta(t, \theta)}. \tag{15}$$

The similarity variable η allows to transform equation (12) into

$$D\frac{\mathrm{d}^2 c}{\mathrm{d}\eta^2} + \eta\frac{\mathrm{d}c}{\mathrm{d}\eta}\left\{\frac{1}{2}\frac{\partial \delta^2}{\partial t} - \tfrac{2}{3}\left[(At^{-\frac{4}{3}} + Bt^{-\frac{3}{2}})\cos\theta - t^{-1}\right]\delta^2 - \tfrac{1}{6}(At^{-\frac{4}{3}} + Bt^{-\frac{3}{2}})\sin\theta\frac{\partial \delta^2}{\partial \theta}\right\} = 0. \tag{16}$$

Equation (16) is compatible with the assumption that the concentration c depends only on η. Indeed, putting

$$\frac{1}{2}\frac{\partial \delta^2}{\partial t} - \tfrac{2}{3}\left[(At^{-\frac{4}{3}} + Bt^{-\frac{3}{2}})\cos\theta - t^{-1}\right]\delta^2 - \tfrac{1}{6}(At^{-\frac{4}{3}} + Bt^{-\frac{3}{2}})\sin\theta\frac{\partial \delta^2}{\partial \theta} = \omega D \tag{17}$$

where ω is a proportionality constant, one obtains

$$\frac{\mathrm{d}^2 c}{\mathrm{d}\eta^2} + \omega\eta\frac{\mathrm{d}c}{\mathrm{d}\eta} = 0. \tag{18}$$

For the constant ω the value 2 will be selected.

The solution of equation (18) for the boundary conditions (14) has the form

$$\frac{c - c_0}{c_i - c_0} = 1 - \frac{2}{\sqrt{\pi}}\int_0^{y/\delta} \mathrm{e}^{-s^2}\,\mathrm{d}s. \tag{19}$$

The thickness δ of the diffusion boundary layer is obtained by solving equation (17) for the initial condition

$$\delta = 0 \quad \text{for} \quad t = 0. \tag{20}$$

The initial condition (20) is a consequence of the initial condition $c = c_0$ for $t = 0$. Indeed only if $\delta = 0$ for $t = 0$, equation (16) satisfies the mentioned initial condition.

Denoting

$$\varepsilon = \delta^2, \tag{21}$$

equation (17) may be written as

$$\frac{\partial \varepsilon}{\partial t} - \tfrac{4}{3}\left[(At^{-\frac{4}{3}} + Bt^{-\frac{3}{2}})\cos\theta - t^{-1}\right]\varepsilon - \tfrac{1}{3}(At^{-\frac{4}{3}} + Bt^{-\frac{3}{2}})\sin\theta\frac{\partial \varepsilon}{\partial \theta} = 4D. \tag{22}$$

The characteristic system which may be attached to equation (22) may be written as:

$$\frac{\mathrm{d}t}{1} = -\frac{3\mathrm{d}\theta}{\sin\theta(At^{-\frac{4}{3}} + Bt^{-\frac{3}{2}})} = \frac{\mathrm{d}\varepsilon}{4D + \tfrac{4}{3}\left[(At^{-\frac{4}{3}} + Bt^{-\frac{3}{2}})\cos\theta - t^{-1}\right]\varepsilon}. \tag{23}$$

From the first equation one obtains

$$3\ln\tan\frac{\theta}{2} - 3At^{-\frac{1}{3}} - 2Bt^{-\frac{1}{2}} = \alpha. \tag{24}$$

As the second equation it will be used (25)

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}\varepsilon}{4D + \frac{4}{3}[(At^{-\frac{4}{3}} + Bt^{-\frac{3}{2}})\cos\theta - t^{-1}]\,\varepsilon}. \tag{25}$$

Because

$$\cos\theta = \frac{1 - \tan^2\dfrac{\theta}{2}}{1 + \tan^2\dfrac{\theta}{2}} \equiv \frac{1 - \exp\left(2\ln\tan\dfrac{\theta}{2}\right)}{1 + \exp\left(2\ln\tan\dfrac{\theta}{2}\right)}$$

equation (24) allows to write

$$\cos\theta = \frac{1 - \exp(\frac{2}{3}\alpha + 2At^{-\frac{1}{3}} + \frac{4}{3}Bt^{-\frac{1}{2}})}{1 + \exp(\frac{2}{3}\alpha + 2At^{-\frac{1}{3}} + \frac{4}{3}Bt^{-\frac{1}{2}})} \tag{26}$$

and equation (25) becomes

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} - \frac{4}{3}\left[(At^{-\frac{4}{3}} + Bt^{-\frac{3}{2}})\frac{1 - \exp(\frac{2}{3}\alpha + 2At^{-\frac{1}{3}} + \frac{4}{3}Bt^{-\frac{1}{2}})}{1 + \exp(\frac{2}{3}\alpha + 2At^{-\frac{1}{3}} + \frac{4}{3}Bt^{-\frac{1}{2}})} - t^{-1}\right]\varepsilon - 4D = 0. \tag{27}$$

by integration one obtains

$$\varepsilon = \left(\exp\left\{\frac{4}{3}\int^{t}\left[(As^{-\frac{4}{3}} + Bs^{-\frac{3}{2}})\frac{1 - \exp(\frac{2}{3}\alpha + 2As^{-\frac{1}{3}} + \frac{4}{3}Bs^{-\frac{1}{2}})}{1 + \exp(\frac{2}{3}\alpha + 2As^{-\frac{1}{3}} + \frac{4}{3}Bs^{-\frac{1}{2}})} - s^{-1}\right]\mathrm{d}s\right\}\right)$$

$$\times\left(E + 4D\int^{t}\left[\exp\left\{-\frac{4}{3}\int^{p}\left[(As^{-\frac{4}{3}} + Bs^{-\frac{3}{2}})\frac{1 - \exp(\frac{2}{3}\alpha + 2As^{-\frac{1}{3}} + \frac{4}{3}Bs^{-\frac{1}{2}})}{1 + \exp(\frac{2}{3}\alpha + 2As^{-\frac{1}{3}} + \frac{4}{3}Bs^{-\frac{1}{2}})} - s^{-1}\right]\mathrm{d}s\right\}\right]\mathrm{d}p\right). \tag{28}$$

The general solutions of equation (22) has the form

$$E = F(\alpha).$$

The form of the function F may be determined by takıng into account the initial condition (20). In this manner it results that

$$F(\alpha) = -4D\int^{0}\left(\exp\left\{-\frac{4}{3}\int^{p}\left[(As^{-\frac{4}{3}} + Bs^{-\frac{3}{2}})\frac{1 - \exp(\frac{2}{3}\alpha + 2As^{-\frac{1}{3}} + \frac{4}{3}Bs^{-\frac{1}{2}})}{1 + \exp(\frac{2}{3}\alpha + 2As^{-\frac{1}{3}} + \frac{4}{3}Bs^{-\frac{1}{2}})} - s^{-1}\right]\mathrm{d}s\right\}\right)\mathrm{d}p$$

and consequently that

$$\varepsilon = 4D\int_{0}^{t}\left(\exp\left\{-\frac{4}{3}\int_{t}^{p}\left[(As^{-\frac{4}{3}} + Bs^{-\frac{3}{2}})\frac{1 - \tan^2(\theta/2)\exp(-2At^{-\frac{1}{3}} - \frac{4}{3}Bt^{-\frac{1}{2}} + 2As^{-\frac{1}{3}} + \frac{4}{3}Bs^{-\frac{1}{2}})}{1 + \tan^2(\theta/2)\exp(-2At^{-\frac{1}{3}} - \frac{4}{3}Bt^{-\frac{1}{2}} + 2As^{-\frac{1}{3}} + \frac{4}{3}Bs^{-\frac{1}{2}})} - s^{-1}\right]\mathrm{d}s\right\}\right)\mathrm{d}p. \tag{29}$$

Performing the integrals from the exponent, one gets

$$\varepsilon = 4Dt^{-\frac{4}{3}} \frac{[1 + \tan^2(\theta/2)]^4}{\exp 2(At^{-\frac{1}{3}} + \frac{4}{3}Bt^{-\frac{1}{2}})} \times \int_0^t p^{\frac{4}{3}} \frac{\exp 2(2Ap^{-\frac{1}{3}} + \frac{4}{3}Bp^{-\frac{1}{2}})\,\mathrm{d}p}{\{1 + \tan^2(\theta/2)\exp[2A(p^{-\frac{1}{3}} - t^{-\frac{1}{3}}) + \frac{4}{3}B(p^{-\frac{1}{2}} - t^{-\frac{1}{2}})]\}^4}. \tag{30}$$

The integral from equation (30) can be carried out probably only numerically. Since the equation used above for the velocity distribution are valid only for small values of the Reynolds number, and therefore for small values of the quantities A and B, we shall perform the integration by expansion in series of the integrand by means of a successive integration by parts. This procedure leads to

$$\begin{aligned}\varepsilon = 4Dt^{-\frac{4}{3}} \frac{[1 + \tan^2(\theta/2)]^4}{\exp 2(2At^{-\frac{1}{3}} + \frac{4}{3}Bt^{-\frac{1}{2}})} \Bigg[&\frac{3}{7} t^{\frac{7}{3}} \frac{\exp 2(2At^{-\frac{1}{3}} + \frac{4}{3}Bt^{-\frac{1}{2}})}{[1 + \tan^2(\theta/2)]^4} + \frac{4}{7}(\tfrac{1}{2}At^2 + \tfrac{6}{11}Bt^{\frac{11}{6}}) \\ &\times \frac{[\exp 2(2At^{-\frac{1}{3}} + \frac{4}{3}Bt^{-\frac{1}{2}})][1 - \tan^2(\theta/2)]}{[1 + \tan^2(\theta/2)]^5} + \frac{16}{21}(\tfrac{3}{10}A^2t^{\frac{5}{3}} + \tfrac{23}{33}ABt^{\frac{3}{2}} + \tfrac{9}{22}B^2t^{\frac{4}{3}}) \\ &\times \frac{[\exp 2(2At^{-\frac{1}{3}} + \frac{4}{3}Bt^{-\frac{1}{2}})][1 - 3\tan^2(\theta/2) + \tan^4(\theta/2)]}{[1 + \tan^2(\theta/2)]^6} + \frac{32}{63}\int_0^t (\tfrac{3}{10}A^2p^{\frac{5}{3}} + \tfrac{23}{33}ABp^{\frac{3}{2}} + \tfrac{9}{22}B^2p^{\frac{4}{3}}) \\ &\times (Ap^{-\frac{4}{3}} + Bp^{-\frac{3}{2}}) \frac{\{1 - \tan^2(\theta/2)\exp[2A(p^{-\frac{1}{3}} - t^{-\frac{1}{3}}) + \frac{4}{3}B(p^{-\frac{1}{2}} - t^{-\frac{1}{2}})]\}}{\{1 + \tan^2(\theta/2)\exp[2A(p^{-\frac{1}{3}} - t^{-\frac{1}{3}}) + \frac{4}{3}B(p^{-\frac{1}{2}} - t^{-\frac{1}{2}})]\}} \\ &\times \Big\{2 + 11\tan^2\frac{\theta}{2}\exp[2A(p^{-\frac{1}{3}} - t^{-\frac{1}{3}}) + \tfrac{4}{3}B(p^{-\frac{1}{2}} - t^{-\frac{1}{2}})] \\ &\qquad + 2\tan^4\frac{\theta}{2}\exp 2[2A(p^{-\frac{1}{3}} - t^{-\frac{1}{3}}) + \tfrac{4}{3}B(p^{-\frac{1}{2}} - t^{-\frac{1}{2}})]\Big\}\mathrm{d}p\Bigg].\end{aligned} \tag{31}$$

Neglecting the last integral as small compared to the other terms, one obtains

$$\varepsilon = \tfrac{12}{7}Dt[1 + \tfrac{2}{3}At^{-\frac{1}{3}}\cos\theta + \tfrac{8}{11}Bt^{-\frac{1}{2}}\cos\theta - \tfrac{2}{3}A^2t^{-\frac{2}{3}}(\tfrac{1}{5} - \cos^2\theta) + \ldots]. \tag{32}$$

Since

$$N_\theta = -D\left(\frac{\partial c}{\partial y}\right)_{y=0} = \frac{2}{\sqrt{\pi}}\frac{D}{\delta}(c_0 - c_i) \tag{33}$$

one gets

$$N_\theta = \left(\frac{7D}{3\pi t}\right)^{\frac{1}{2}}(c_0 - c_i)\left[1 - \frac{Re\cos\theta}{3(2 + 5\beta)} - \frac{56}{55}\frac{(\sqrt{3})\,Re^{\frac{1}{2}}\cos\theta}{(2 + 3\beta)} + \frac{Re^2}{6(2 + 5\beta)^2}\left(\frac{2}{5} - \cos^2\theta\right) + \ldots\right]. \tag{34}$$

For the average value of the mass flux in the continuous phase there results

$$N = \left(\frac{7D}{3\pi t}\right)^{\frac{1}{2}}(c_0 - c_i)\left[1 + \frac{Re^2}{90(2 + 5\beta)^2} + \ldots\right]. \tag{35}$$

The first term in equation (35) represents Ilkovic's approximation, while the others represent correction to his result. The obtained equation is the same as the one obtained by Golub and Krîlov, but was deduced here by means of a more simple method. Besides, the present procedure may be used also when the contribution of the tangential velocity component is important (large Reynolds numbers), while the perturbation technique used by the mentioned authors is restricted to small values of the tangential velocity. Unfortunately for the time being no velocity distribution for a growing drop at large Reynolds number exists.

It may be noted that the method may be extended to all cases in which the time dependence of the radius is known; it may be also applied to the cases in which the growth of the bubble is determined by the heat transfer itself. In the last case one obtains for the radius a non-linear integro-differential equation which can be solved only numerically.

REFERENCES

1. H. K. Forster and N. Zuber, Dynamics of vapor bubbles and boiling heat transfer, *AIChE Jl* **1**, 531 (1955).
2. M. Plesset and S. A. Zwick, The growth of vapour bubbles in super-heated liquids, *J. Appl. Phys.* **25**, 493 (1954).
3. L. E. Scriven, On the dynamics of phase growth, *Chem. Engng Sci.* **10**, 1 (1959).
4. D. Ilković, Polarographic studies with the dropping mercury cathode. *Coll. Trav. Chim. Tchecoslov.* **6**, 498 (1934).
5. W. J. Beek and H. Kramers, Mass transfer with a change in interfacial area, *Chem. Engng Sci.* **16**, 909 (1962).
6. K. Golub and V. S. Crîlov, On the convective diffusion theory for a growing drop, *Isv. Akad. Nauk SSSR Meh. Jidkosti. Gaza*, p. 58 nr. 1 (1968).
7. E. Ruckenstein, A generalized penetration theory for unsteady convective mass transfer, *Chem. Engng Sci.* **23**, 363 (1968).

Résumé—Une transformation de similitude est employée pour résoudre le problème du transport de masse vers une goutte ou une bulle croissante lorsqu'on tient compte non seulement de la composante radiale de la vitesse (comme dans des articles antérieurs) mais aussi de la composante tangentielle de la vitesse. La méthode peut être employée lorsque la croissance est due soit au transport de chaleur lui-même, soit à l'écoulement d'un liquide conduit par un capillaire.

Zusammenfassung—Mit Hilfe einer Ähnlichkeitsbetrachtung wird das Problem des Massentransports zu einem wachsenden Tropfen oder einer Blase gelöst, wobei nicht nur die radiale Komponente der Geschwindigkeit berücksichtigt wird (wie in früheren Arbeiten) sondern auch die Tangential-Komponente. Die Methode kann angewendet werden, wenn das Wachstum aufgrund der Wärmeübertragung selbst erfolgt oder wenn es aufgrund der Zuströmung von einer Flüssigkeit in einer Kapillaren erfolgt.

Аннотация—Метод может быть использован в случае, когда рост капли происходит в результате теплообмена или за счет течения жидкости, подаваемой через капилляр. Задача массопереноса капли, увеличивающейся в размерах, решается методом подобных преобразований. При решении кроме радиальной составляющей (см. предыдущие статьи) учитывается и тангенциальная составляющая.